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Response of welded thermoelastic solids to the rapid motion of thermomechanical sources parallel to the interface

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Abstract

The analysis of rapidly-moving thermomechanical surface sources is extended to the study of buried thermomechanical sources that move parallel to the interface of two welded dissimilar thermoelastic half-spaces at a constant subcritical speed. The sources are manifest as body force line loads in the coupled equations of thermoelasticity, and a 2-D steady-state situation is treated. Exact integral transform solutions are obtained, and expressions for the displacements and temperature changes are generated by analytical inversion of robust asymptotic versions of the transforms.

These expressions show that thermoelastic coupling effects increase with source speed, and that the thermal source is always manifest in combination with a component of the mechanical source, i.e. an effective thermal source term exists. The expressions also exhibit component functions that are in effect hybrids of functions that are seen in purely thermal and isothermal elastic solutions.

The critical source speed is defined as the minimum of the two asymptotic thermoelastic Rayleigh speeds in the half-spaces and, when it exists, the asymptotic thermoelastic Stoneley speed. Exact expressions for these speeds are given, and used to present some typical values. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

Brock and Georgiadis (1997) and Brock et al. (1997) have treated rapid motion by thermomechanical loads over the surfaces of thermoelastic half-spaces. These half-spaces are modeled by the coupled forms of the momentum balance and thermal diffusion equations (Chadwick, 1960; Boley and Weiner, 1985). The coupled thermoelastic studies demonstrate that, due to the existence of a small, i.e. $O(10^{-4})$ μm, thermoelastic characteristic length in the equations, robust asymptotic

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solutions can be obtained analytically. Moreover, these solutions show that the influence of thermoelastic coupling is noticeable, especially at higher load speeds.

This work extends these efforts by considering buried thermomechanical sources that translate parallel to the interfaces of dissimilar coupled thermoelastic half-spaces that are rigidly welded. As in the previous work, the sources are line loads moving at a constant speed, so that a 2-D steady-state analysis can be performed in terms of half-planes. In this case, only sub-critical speeds are considered.

In the next section, the problem is formulated, and addressed by transform methods. From exact transform solutions, robust asymptotic analytical expressions for the displacements and temperature changes are then obtained by inversion. These expressions show the same types of thermoelastic coupling effects seen—especially at high source speeds—by Brock and Georgiadis (1997) and Brock et al. (1997). In particular, thermoelastic constants influence both the coefficients and arguments of various functions that constitute the expressions. In the present work, moreover, some of the functions themselves are seen to be, in essence, hybrids of responses seen in purely mechanical and thermal analyses. It is also found that the body force term in the coupled thermal diffusion equation is always manifest in solution expressions in a linear combination with the body force component of the momentum balance equation that lies parallel to the source motion direction. That is, the displacements and temperature change depend on pure mechanical loading and an effective thermal loading.

2. Problem formulation

Consider two half-spaces of dissimilar isotropic homogeneous linearly thermoelastic materials that are rigidly welded together over the $x'z'$-plane, where (x', y', z') are Cartesian coordinates. The half-space $y' > 0$ is denoted as solid 1, and its field variables and thermoelastic properties carry the subscript 1; analogously, solid 2 comprises the half-space $y' < 0$, and its field variables and thermoelastic properties carry the subscript 2. Both solids are initially at rest at the uniform (absolute) temperature T_0. Then constant thermomechanical body forces are induced at time $t = 0$ along an infinite line that lies parallel to the z'-axis and translates in the positive x'-direction at a subcritical constant speed v. No generality is lost by fixing the line's path of travel at a distance d from the interface in solid 1 ($y' > 0$).

For convenience the moving Cartesian system

$$x = x' - vt, \quad y = y', \quad z = z' \tag{1}$$

is introduced so that the line of sources is always located at $(x, y) = (0, d)$. No dependence on z is expected, so that solids 1 and 2 can be treated as the half-planes $y > 0$ and $y < 0$, respectively, and the relevant problem geometry can be represented as Fig. 1. There the constants (B_x, B_y) are the x- and y-components of the mechanical body force, while the constant B_q represents the scalar thermal body force. In terms of the moving coordinate system, time derivatives take the form (Bowen, 1989)

$$\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} \tag{2}$$

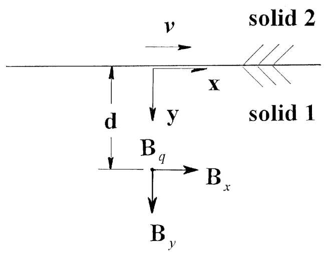

Fig. 1. Thermomechanical sources moving near interface of welded solids.

where the first operator is to be taken in the moving system, and the second operator corrects for the motion of the system. If, as in this analysis, the steady-state is of interest, then the first operator in (2) can be neglected, and all the field variables treated as functions of (x, y) only. Then the governing steady-state coupled thermoelastic field equations for solid 1 ($y > 0$) can be obtained from, respectively, the momentum balance and thermal diffusion laws (Chadwick, 1960; Boley and Weiner, 1985; Achenbach, 1973) as

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla\Delta-\chi_0(3\lambda+2\mu)\nabla\theta-\rho v^2\frac{\partial^2\mathbf{u}}{\partial x^2}+\mathbf{B}\delta(x)\delta(y-d)=0 \tag{3a}$$

$$k\nabla^2\theta+v\frac{\partial}{\partial x}[c_v\rho\theta+\chi_0(3\lambda+2\mu)T_0\Delta]+B_1\delta(x)\delta(y-d)=0 \tag{3b}$$

where the subscript 1 is understood. In (3) δ is the Dirac function, $\mathbf{u}(x,y)=(u_x,u_y)$ is the displacement vector, $\theta(x,y)$ is the change in temperature from the value T_0, $\mathbf{B}=(B_x,B_y)$, Δ is the 2-D dilatation, (λ,μ) are the Lamé constants, ρ is the mass density, and (k,χ_0,c_v) are, respectively, the thermal conductivity, coefficient of expansion and specific heat. Introduction of the thermoelastic parameters

$$v_d=\sqrt{\frac{\lambda+2\mu}{\rho}},\quad v_r=\sqrt{\frac{\mu}{\rho}},\quad s_d=\frac{1}{v_d},\quad s_r=\frac{1}{v_r} \tag{4a}$$

$$m=\frac{v_d}{v_r},\quad c=\frac{v}{v_d} \tag{4b}$$

$$\chi=\chi_0(4-3m^2),\quad h=\frac{kv_r}{\mu m c_v},\quad \varepsilon=\frac{T_0}{c_v}\left(\frac{\chi}{m}v_r\right)^2 \tag{4c}$$

allows the governing field eqns (3) to be written as

$$\nabla^2\mathbf{u}+(m^2-1)\nabla\Delta+\chi\nabla\theta-m^2c^2\frac{\partial^2\mathbf{u}}{\partial x^2}+\frac{1}{\mu}\mathbf{B}\delta(x)\delta(y-d)=0 \tag{5a}$$

$$h\nabla^2\theta + c\frac{\partial}{\partial x}\left(\theta - \frac{m^2\varepsilon}{\chi}\Delta\right) + \frac{B_q}{\rho c_v v_d}\delta(x)\delta(y-d) = 0. \tag{5b}$$

In terms of (4), the associated constitutive equations (Achenbach, 1973) give the 2-D stress tensor

$$\frac{1}{\mu}\underline{\underline{\sigma}} = \nabla\mathbf{u} + \mathbf{u}\nabla + [(m^2-2)\Delta + \chi\theta]\underline{\underline{I}} \tag{6}$$

where $\underline{\underline{I}}$ is the identity tensor. In (4) the quantities (v_d, v_r) are the isothermal dilatational and rotational wave speeds, respectively, and (s_d, s_r) are the corresponding slownesses, so that $(m, \chi, c, \varepsilon)$ are dimensionless, and h has the dimensions of length. In particular, c is the dimensionless source speed, while h is the thermoelastic characteristic length and ε is the well-known (Chadwick, 1960) dimensionless coupling constant. It can be shown (Sokolnikoff, 1956; Chadwick, 1960; Brock, 1992) for many materials that

$$m > \sqrt{2}, \quad \varepsilon = O(10^{-2}), \quad h = O(10^{-4})\ \mu\text{m}. \tag{7}$$

The order of ε in particular is often used to justify dropping the corresponding term from (5b), thereby uncoupling this thermal diffusion equation from the momentum balance eqn (5a) (Boley and Weiner, 1985). Equations analogous to (4)–(7) exist for solid 2 ($y < 0$), with subscript 2 understood, but no $(B_q, \mathbf{B})$-terms present. For now, subcritical v will be taken to mean

$$v < \min(v_{r1}, v_{r2}). \tag{8}$$

That is, the source speed does not exceed the lowest rotational wave speed among the two solids. The welded bond between the two solids implies continuity of displacement, tractions, temperature change and heat flux along the interface, i.e.

$$\mathbf{u}_1 - \mathbf{u}_2 = \sigma_{xy1} - \sigma_{xy2} = \sigma_{y1} - \sigma_{y2} = \theta_1 - \theta_2 = \frac{\partial\theta_1}{\partial y} - \frac{\partial\theta_2}{\partial y} = 0. \tag{9}$$

In addition, experience with 2-D versions of the purely mechanical Kelvin problem (Sokolnikoff, 1956) suggest that $(\mathbf{u}_1, \theta_1)$ and $(\mathbf{u}_2, \theta_2)$ should behave no worse than logarithmically as $\sqrt{x^2+y^2} \to \infty$, and should be continuous for all $y > 0$, $x \neq 0$ and $y < 0$, respectively.

The problem formulation defined by these conditions and (4)–(9) is tractable, but the Dirac function nature of the non-homogeneous terms in (5a,b) for $y > 0$ suggest (Stakgold, 1967) the following alternative formulation: if the solution fields $(\mathbf{u}_1^{(-)}, \theta_1^{(-)})$ and $(\mathbf{u}_1^{(+)}, \theta_1^{(+)})$ are associated with, respectively, the regions $0 < y < d$ and $y > d$, the aforementioned continuity combined with integration in y across the line $y = d$ gives the matching/jump conditions

$$\mathbf{u}_1^{(+)} - \mathbf{u}_1^{(-)} = \theta_1^{(+)} - \theta_1^{(-)} = 0 \tag{10a}$$

$$\frac{\partial u_1^{(+)}}{\partial y} - \frac{\partial u_1^{(-)}}{\partial y} + (m_1^2-1)\frac{\partial}{\partial x}(v_1^{(+)} - v_1^{(-)}) = -\frac{B_y}{\mu_1}\delta(x) \tag{10b}$$

$$\frac{\partial v_1^{(+)}}{\partial y} - \frac{\partial v_1^{(-)}}{\partial y} + (m_1^2-1)\left[\frac{\partial}{\partial x}(u_1^{(+)} - u_1^{(-)}) + \frac{\partial v_1^{(+)}}{\partial y} - \frac{\partial v_1^{(-)}}{\partial y}\right] + \chi_1(\theta_1^{(+)} - \theta_1^{(-)}) = -\frac{B_y}{\mu_1}\delta(x) \tag{10c}$$

$$h_1\left(\frac{\partial\theta_1^{(+)}}{\partial y}-\frac{\partial\theta_1^{(-)}}{\partial y}\right)+\left(\frac{m^2\varepsilon c}{\chi}\right)_1\frac{\partial}{\partial x}(v_1^{(+)}-v_1^{(-)}) = -\frac{B_q}{(\rho c_v v_d)_1}\delta(x) \tag{10d}$$

along $y = d$ for the two fields. Equation (9) then gives the matching conditions for the fields $(\mathbf{u}_1^{(-)}, \theta_1^{(-)})$ and $(\mathbf{u}_2, \theta_2)$. In this manner, an alternative formulation of three layers $(y < 0, 0 < y < d, y > d)$ arises, but one in which the governing field equations are now all homogeneous, i.e. the form of (5a,b) less $(\mathbf{B}, B_q)$ holds for all three layers. This procedure avoids the necessity of finding particular solutions. The alternative problem formulation is addressed by transform methods in the next section.

3. Transform solution

To solve the problem formulated above, the bilateral Laplace transform (van der Pol and Bremmer, 1950) and its inverse operation

$$f^* = \int_{-\infty}^{\infty} f(x)\,e^{-px}\,\mathrm{d}x, \quad f(x) = \frac{1}{2\pi i}\int f^*\,e^{px}\,\mathrm{d}p \tag{11a,b}$$

are introduced. Here p is, in general, complex and integration in (11b) is taken along the Bromwich contour. Application of (11a) to (5) with $(\mathbf{B}, B_q)$ dropped out gives the following set of general transform solutions:

$$\begin{bmatrix} u_x^* \\ \dfrac{1}{p^2}\theta^* \\ \dfrac{1}{\mu p}\sigma_y^* \end{bmatrix} = \begin{bmatrix} -p & -p & 1 \\ \omega_+ & \omega_- & 0 \\ -Kp & -Kp & -2 \end{bmatrix}\begin{bmatrix} A_+\,e^{\alpha_+ y}+A_-\,e^{-\alpha_+ y} \\ B_+\,e^{\alpha_- y}+B_-\,e^{-\alpha_- y} \\ C_+\,e^{\beta y}+C_-\,e^{-\beta y} \end{bmatrix} \tag{12a}$$

$$\begin{bmatrix} u_y^* \\ \dfrac{1}{p^2}\dfrac{\partial\theta^*}{\partial y} \\ \dfrac{1}{\mu p}\sigma_{xy}^* \end{bmatrix} = \begin{bmatrix} -1 & -1 & -p \\ \omega_+ & \omega_- & 0 \\ -2 & -2 & Kp \end{bmatrix}\begin{bmatrix} \alpha_+(A_+\,e^{\alpha_+ y}-A_-\,e^{-\alpha_+ y}) \\ \alpha_-(B_+\,e^{\alpha_- y}-B_-\,e^{-\alpha_- y}) \\ \dfrac{1}{\beta}(C_+\,e^{\beta y}-C_-\,e^{-\beta y}) \end{bmatrix}. \tag{12b}$$

Here the coefficients $(A_\pm, B_\pm, C_\pm)$ are arbitrary functions of p and

$$\alpha_\pm = a_\pm\sqrt{-p^2}, \quad \beta = b\sqrt{-p^2}, \quad \omega_\pm = \frac{m^2}{\chi}(1-c^2-a_\pm^2) \tag{13a}$$

$$a_\pm = \sqrt{1+\frac{c}{p}(\tau_+\pm\tau_-)^2}, \quad b = \sqrt{1-m^2c^2}, \quad K = m^2c^2-2 \tag{13b}$$

$$2\tau_{\pm} = \sqrt{\left(\sqrt{-cp} \pm \frac{1}{\sqrt{h}}\right)^2 + \frac{\varepsilon}{h}}, \quad \omega_+\omega_- = \frac{m^4 c^3 \varepsilon}{\chi^2 ph} \tag{13c}$$

where $\mathrm{Re}(\alpha_{\pm}, \beta) \geqslant 0$ in the cut p-plane. It should be noted that restriction (8) guarantees that the constant b is real and positive.

Because there are three layers (12a,b) require that 18 coefficients be determined from (9), (10) and boundedness conditions. The latter in particular require that $(\mathbf{u}_1^{(+)}, \theta_1^{(+)})^*$ and $(\mathbf{u}_2, \theta_2)^*$ not become unbounded as $y \to \infty$ and $y \to -\infty$, respectively. Therefore, in (12a,b) we must have

$$A_{1+}^{(+)} = B_{1+}^{(+)} = C_{1+}^{(+)} = A_{2-} = B_{2-} = C_{2-} = 0. \tag{14}$$

Application of (11a) to the matching conditions (9) and (10) in view of (12) and (14) gives twelve equations that can be solved analytically for the other coefficients. Specifically, we have

$$\begin{bmatrix} A_{1+}^{(-)} \\ B_{1+}^{(-)} \\ C_{1+}^{(-)} \end{bmatrix} = \frac{1}{2(\omega_{1+} - \omega_{1-})p^2} \begin{bmatrix} A_0 \\ B_0 \\ (\omega_{1+} - \omega_{1-})pC_0 \end{bmatrix} \tag{15a}$$

$$\begin{bmatrix} A_{1-}^{(+)} \\ B_{1-}^{(+)} \\ C_{1-}^{(+)} \end{bmatrix} = \frac{1}{2(\omega_{1+} - \omega_{1-})p^2} \begin{bmatrix} A_0' \\ B_0' \\ (\omega_{1+} - \omega_{1-})pC_0' \end{bmatrix} + \begin{bmatrix} A_{1-}^{(-)} \\ B_{1-}^{(-)} \\ C_{1-}^{(-)} \end{bmatrix} \tag{15b}$$

where $(A_{1-}^{(-)}, B_{1-}^{(-)}, C_{1-}^{(-)})$ are given by

$$\begin{bmatrix} A_{1-}^{(-)} \\ B_{1-}^{(-)} \\ C_{1-}^{(-)} \end{bmatrix} = \frac{1}{2\Omega_1} \begin{bmatrix} E_+\omega_{2+} - D_+\omega_{1-} & E_{\mp}\omega_{2-} - D_{\mp}\omega_{1-} & \left(1 + \dfrac{Q}{a_{1+}b_2}\right)\dfrac{\omega_{1-}}{p} \\ D_{\pm}\omega_{1+} - E_{\pm}\omega_{2+} & D_-\omega_{1+} - E_-\omega_{2-} & \left(1 + \dfrac{Q}{a_{1-}b_2}\right)\dfrac{\omega_{1+}}{p} \\ p(Q + a_{2+}b_1) & p(Q + a_{2-}b_1) & \dfrac{b_1}{b_2}P_2 + P_1 \end{bmatrix} \begin{bmatrix} A_{2+} \\ B_{2+} \\ C_{2+} \end{bmatrix} \tag{16}$$

and (A_{2+}, B_{2+}, C_{2+}) are, in turn, defined by

$$\begin{bmatrix} A_{2+} \\ B_{2+} \\ pC_{2+} \end{bmatrix} = \frac{1}{p^2 S} \begin{bmatrix} a_{1+}L_{11} & a_{1-}L_{12} & L_{13} \\ -a_{1+}L_{21} & -a_{1-}L_{22} & L_{23} \\ a_{1+}b_2L_{31} & a_{1-}b_2L_{32} & b_2L_{33} \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ p^2C_0 \end{bmatrix}. \tag{17}$$

The coefficients of the matrix $\mathbf{L}$ are

$$L_{11} = \Omega_1(a_{1-} + a_{2-})(b_1P_2 - b_2P_1)\omega_{2-} + S_-\omega_{1+} \tag{18a}$$

$$L_{12} = \Omega_1(a_{1+} + a_{2-})(b_1P_2 - b_2P_1)\omega_{2-} + S_{\mp}\omega_{1-} \tag{18b}$$

$$L_{13} = (a_{1+} - a_{1-})(P_2Q - P_1a_{2-}b_2)\omega_{1+}\omega_{1-} + (a_{1-} + a_{2-})(Q - a_{1+}b_2)\omega_{1-}\omega_{2-} - (a_{1+} + a_{2-})(Q - a_{1-}b_2)\omega_{1+}\omega_{2-} \tag{18c}$$

$$L_{21} = \Omega_1(a_{1-}+a_{2+})(b_1P_2-b_2P_1)\omega_{2+}+S_{\pm}\omega_{1+} \tag{18d}$$

$$L_{22} = \Omega_1(a_{1+}+a_{2+})(b_1P_2-b_2P_1)\omega_{2+}+S_{+}\omega_{1-} \tag{18e}$$

$$\begin{aligned} L_{23} &= (a_{1-}-a_{1+})(P_2Q-P_1a_{2+}b_2)\omega_{1+}\omega_{1-} \\ &\quad +(a_{1+}+a_{2+})(Q-a_{1-}b_2)\omega_{1+}\omega_{2+}-(a_{1-}+a_{2+})(Q-a_{1+}b_2)\omega_{1-}\omega_{2+} \end{aligned} \tag{18f}$$

$$\begin{aligned} L_{31} &= \Omega_1(a_{2+}+a_{1-})(Q-a_{2-}b_1)\omega_{2+}-\Omega_1(a_{1-}+a_{2-})(Q-a_{2+}b_1)\omega_{2-} \\ &\quad +(a_{2+}-a_{2-})(P_1Q-P_2a_{1-}b_1)\omega_{1+} \end{aligned} \tag{18g}$$

$$\begin{aligned} L_{32} &= \Omega_1(a_{1+}+a_{2+})(Q-a_{2-}b_1)\omega_{2+}-\Omega_1(a_{1+}+a_{2-})(Q-a_{2+}b_1)\omega_{2-} \\ &\quad +(a_{2+}-a_{2-})(P_1Q-P_2a_{1+}b_1)\omega_{1-} \end{aligned} \tag{18h}$$

$$\begin{aligned} L_{33} &= (a_{1+}-a_{1-})(a_{2+}-a_{2-})(\omega_{2+}\omega_{2-}-P_1P_2\omega_{1+}\omega_{1-}) \\ &+\Omega_1(a_{1-}+a_{2+})(P_2a_{1+}-P_1a_{2-})\omega_{1-}\omega_{2+}+\Omega_1(a_{1+}+a_{2-})(P_2a_{1-}-P_1a_{2+})\omega_{1+}\omega_{2-} \\ &-\Omega_1(a_{1+}+a_{2+})(P_2a_{1-}-P_1a_{2-})\omega_{1+}\omega_{2+}-\Omega_1(a_{1-}+a_{2-})(P_2a_{1+}-P_1a_{2+})\omega_{1-}\omega_{2-}. \end{aligned} \tag{18i}$$

In these formulae, the thermomechanical source terms are manifest in the quantities

$$\begin{bmatrix} A_0\, e^{\alpha_{1+}d} \\ B_0\, e^{\alpha_{1-}d} \\ C_0\, e^{\beta_1 d} \end{bmatrix} = \begin{bmatrix} -\dfrac{\omega_{1-}}{\alpha_{1+}} & \dfrac{\omega_{1-}}{p} & -\dfrac{1}{\alpha_{1+}} \\ \dfrac{\omega_{1+}}{\alpha_{1-}} & -\dfrac{\omega_{1+}}{p} & \dfrac{1}{\alpha_{1-}} \\ \dfrac{\beta_1}{p} & -1 & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_q \end{bmatrix} \tag{19a}$$

$$\begin{bmatrix} A'_0\, e^{-\alpha_{1+}d} \\ B'_0\, e^{-\alpha_{1-}d} \\ C'_0\, e^{-\beta_1 d} \end{bmatrix} = \begin{bmatrix} -\dfrac{\omega_{1-}}{\alpha_{1+}} & -\dfrac{\omega_{1-}}{p} & -\dfrac{1}{\alpha_{1+}} \\ -\dfrac{\omega_{1+}}{\alpha_{1-}} & \dfrac{\omega_{1+}}{p} & \dfrac{1}{\alpha_{1-}} \\ \dfrac{\beta_1}{p} & 1 & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_q \end{bmatrix} \tag{19b}$$

where

$$b_x = \frac{B_x}{(\mu m^2c^2)_1}, \quad b_y = \frac{B_y}{(\mu m^2c^2)_1}, \quad b_q = \left(\frac{h}{\rho c_v v_d}\right)_1 B_q - \left(\frac{\varepsilon}{\mu\chi hc}\right)_1 B_x. \tag{20a–c}$$

The other previously-undefined quantities in (15)–(18) are

$$(E_+, E_-, E_\pm, E_\mp) = \frac{\Omega_1}{\omega_{1+}-\omega_{1-}}\left[1-\left(\frac{a_{2+}}{a_{1+}}, \frac{a_{2-}}{a_{1-}}, \frac{a_{2+}}{a_{1-}}, \frac{a_{2-}}{a_{1+}}\right)\right] \tag{21a}$$

$$(D_+, D_-, D_\pm, D_\mp) = \frac{1}{\omega_{1+} - \omega_{1-}} \left[P_1 + P_2 \left(\frac{a_{2+}}{a_{1+}}, \frac{a_{2-}}{a_{1-}}, \frac{a_{2+}}{a_{1-}}, \frac{a_{2-}}{a_{1+}} \right) \right] \tag{21b}$$

and

$$\begin{aligned} S &= (b_1 P_2 - b_2 P_1)(a_{1+} - a_{1-})(a_{2+} - a_{2-})(\Omega_1 \omega_{2+} \omega_{2-} - \Omega_2 \omega_{1+} \omega_{1-}) \\ &+ (a_{1-} + a_{2-}) \omega_{1-} \omega_{2-} S_+ + (a_{1+} + a_{2+}) \omega_{1+} \omega_{2+} S_- \\ &- (a_{1+} + a_{2-}) \omega_{1+} \omega_{2-} S_\pm - (a_{1-} + a_{2+}) \omega_{1-} \omega_{2+} S_\mp \end{aligned} \tag{22a}$$

$$\begin{bmatrix} S_+ \\ S_- \\ S_\pm \\ S_\mp \end{bmatrix} = Q^2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a_{1+}a_{2+} & a_{1+}b_2 + a_{2+}b_1 & a_{1+} & a_{2+} \\ a_{1-}a_{2-} & a_{1-}b_2 + a_{2-}b_1 & a_{1-} & a_{2-} \\ a_{1-}a_{2+} & a_{1-}b_2 + a_{2+}b_1 & a_{1-} & a_{2+} \\ a_{1+}a_{2-} & a_{1+}b_2 + a_{2-}b_1 & a_{1+} & a_{2-} \end{bmatrix} \begin{bmatrix} b_1 b_2 \\ \Omega_1 \Omega_2 \\ -b_1 P_2^2 \\ -b_2 P_1^2 \end{bmatrix} \tag{22b}$$

where

$$P_1 = 1 - \Omega_1, \quad P_2 = 1 - \Omega_2, \quad Q = 1 - \Omega_1 - \Omega_2, \quad \Omega_1 = \frac{(\mu m^2 c^2)_1}{2(\mu_1 - \mu_2)}, \quad \Omega_2 = \frac{(\mu m^2 c^2)_2}{2(\mu_2 - \mu)}. \tag{23}$$

It should be noted that the common denominator term S is the steady-state coupled thermoelastic Stoneley function (Stoneley, 1924) for the two welded solids. Indeed, the constituent functions $(S_+, S_-, S_\pm, S_\mp)$ have the same form as the classical isothermal Stoneley function (Cagniard, 1962). Examination of (20c) shows, moreover, the coupling behavior between the thermal source term B_q and the mechanical source term B_x that was noted at the outset. This coupling produces an effective thermal source insofar as solution response is concerned.

With (14)–(17) in hand (12a,b) give the complete transform solutions for the fields $(\mathbf{u}_1^{(\pm)}, \theta_1^{(\pm)})$ and $(\mathbf{u}_2, \theta_2)$ in the layers $(0 < y < d, y > d)$ and $(y < 0)$, respectively. These solutions are exact, but largely due to the p-dependence of the quantities $(a_{1\pm}, a_{2\pm})$ [see (13b,c)], their inversion by means of (11b) must be carried out numerically. While several efficient numerical schemes are available, e.g. Duffy (1993), we use here an asymptotic form of the transform solutions are quite robust, yet which allow inversions to be performed analytically.

4. Asymptotic transform results

It is known (van der Pol and Bremmer, 1950) that asymptotic forms of the bilateral Laplace transform valid for small $|pL|$ give inversions that are valid for large $|x/L|$, where L is any finite scaling length. For the previous results, therefore, we substitute (14)–(17) into (12a,b) and treat the expressions created as linear combinations of the loading parameters (b_x, b_y, b_q). The coefficients of each parameter are then expanded for small $|p|$, and only the lowest-order terms preserved. The key step in this operation is that (13) yields the asymptotic forms

$$\alpha_+ = \sqrt{-\lambda p}, \quad \alpha_- = a\sqrt{-p^2}, \quad a_+ = \sqrt{\frac{\lambda}{p}}, \quad a_- = a, \quad \omega_+ = -\frac{\Gamma}{p}, \quad \omega_- = -\gamma \tag{24a}$$

$$a = \sqrt{1-\frac{c^2}{1+\varepsilon}}, \quad \Gamma = \frac{m^2\lambda}{\chi}, \quad \gamma = \frac{m^2 c^2 \varepsilon}{\chi(1+\varepsilon)}, \quad \lambda = \frac{c}{h}(1+\varepsilon) \tag{24b}$$

where it is noted that (7) and (8) guarantee that a is a positive real constant, and $v_d\sqrt{1+\varepsilon}$ is an asymptotic thermoelastic (adiabatic) dilatational wave speed (Achenbach, 1973). However, the forms (24a) are actually only valid for $|ph| \ll 1$, so that in view of the previous observation, the inversions of the asymptotic forms of (12) are valid for $|x/h| \gg 1$. But (7) shows that (h_1, h_2) are of the same very small magnitude order, which implies that the inversions are robust. This result is not affected by the scale of the other characteristic length, d, which appears only in the exponential arguments of the source term functions (A_0, B_0, C_0) and (A'_0, B'_0, C'_0).

Because (a, b, λ) are real and positive, the condition $\mathrm{Re}(\alpha_\pm, \beta) \geqslant 0$ in the cut p-plane which was imposed for boundedness upon (12a,b) requires that the asymptotic forms of both (α_-, β) given by (24a) have the branch cuts $\mathrm{Im}(p) = 0$, $|\mathrm{Re}(p)| > 0$, while α_+ must exhibit the cut $\mathrm{Im}(p) = 0$, $\mathrm{Re}(p) > 0$.

To illustrate these forms, we present those for the displacement and temperature change: for $y < 0$ the asymptotic forms

$$\begin{bmatrix} u^*_{2x} \\ \dfrac{\sqrt{-p}}{\sqrt{p}}u^*_{2y} \\ \dfrac{1}{p}\theta^*_2 \end{bmatrix} = \left(\frac{b_x}{\sqrt{-p^2}}\mathbf{M}_{2x} + \frac{b_y}{p}\mathbf{M}_{2y} + \frac{b_q}{\sqrt{-p^2}}\mathbf{M}_{2\theta}\right)\begin{bmatrix} e^{\sqrt{-\lambda_2 p}y} \\ e^{a_2\sqrt{-p^2}y} \\ e^{b_2\sqrt{-p^2}y} \end{bmatrix} \tag{25a}$$

$$\mathbf{M}_{2n} = \begin{bmatrix} 0 & -B_n & C_n \\ 0 & a_2 B_n & -\dfrac{1}{b_2}C_n \\ -\Gamma_2 A_n & -\gamma_2 B_n & 0 \end{bmatrix}, \quad \mathbf{M}_{2\theta} = \begin{bmatrix} 0 & -B_b & C_b \\ -\sqrt{\lambda_2}A_b & a_2 B_b & -\dfrac{1}{b_2}C_b \\ -\Gamma_2 A_b\sqrt{p} & 0 & 0 \end{bmatrix} \tag{25b,c}$$

hold, where the subscript $n = (x, y)$ in (25b) and

$$\begin{bmatrix} A_x \\ \dfrac{1}{M_{11}}B_x \\ \dfrac{1}{M_{11}}C_x \end{bmatrix} = \frac{1}{D_0}\begin{bmatrix} 0 & M_{13}M_{32}-M_{12}M_{33} & M_{12}M_{23}-M_{13}M_{22} \\ 0 & M_{33} & -M_{23} \\ 0 & -M_{32} & M_{22} \end{bmatrix}\begin{bmatrix} N_{11} \\ N_{21} \\ N_{31} \end{bmatrix} \tag{26a}$$

$$\begin{bmatrix} A_y \\ \dfrac{1}{M_{11}}B_y \\ \dfrac{1}{M_{11}}C_y \end{bmatrix} = \frac{1}{D_0}\begin{bmatrix} \dfrac{D_0}{M_{11}} & M_{13}M_{32}-M_{12}M_{33} & M_{12}M_{23}-M_{13}M_{22} \\ 0 & M_{33} & -M_{23} \\ 0 & -M_{32} & M_{22} \end{bmatrix}\begin{bmatrix} N_{12} \\ N_{22} \\ N_{32} \end{bmatrix} \tag{26b}$$

$$\begin{bmatrix} A_b \\ B_b \\ C_b \end{bmatrix} = \frac{1}{D_0} \begin{bmatrix} \dfrac{D_0}{M_{11}} & 0 & 0 \\ M_{31}M_{23}-M_{21}M_{33} & M_{33} & 0 \\ M_{21}M_{32}-M_{31}M_{22} & -M_{32} & 0 \end{bmatrix} \begin{bmatrix} N_{13} \\ M_{11}N_{23} \\ N_{33} \end{bmatrix} \tag{26c}$$

and the elements of the non-symmetric matrices $\mathbf{M}$ and $\mathbf{N}$ are given by

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -\Gamma_2\Omega_1\left(1+\sqrt{\dfrac{\lambda_2}{\lambda_1}}\right) & \gamma_1 P_2-\gamma_2\Omega_1 & -\gamma_1 \\ -\dfrac{\sqrt{\lambda_2}}{a_1}(\Gamma_1 P_1+\Gamma_2\Omega_1) & \Gamma_1\left(P_2-\dfrac{a_2}{a_1}P_1\right) & -\Gamma_1\left(1-\dfrac{Q}{a_1 b_2}\right) \\ -b_1\sqrt{\lambda_2} & Q-a_2 b_1 & \dfrac{b_1}{b_2}P_2-P_1 \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} N_{11} \\ N_{12} \\ N_{13} \end{bmatrix} = \begin{bmatrix} \dfrac{\gamma_1}{\sqrt{\lambda_1}} \\ -\gamma_1 \\ -\dfrac{1}{\sqrt{\lambda_1}} \end{bmatrix} e^{-\sqrt{-\lambda_1}pd}, \quad \begin{bmatrix} N_{21} \\ N_{22} \\ N_{23} \end{bmatrix} = \begin{bmatrix} \dfrac{\Gamma_1}{a_1} \\ -\Gamma_1 \\ -\dfrac{1}{a_1} \end{bmatrix} e^{-a_1\sqrt{-p^2}d}, \tag{28}$$

$$\begin{bmatrix} N_{31} \\ N_{32} \\ N_{33} \end{bmatrix} = \begin{bmatrix} -b_1 \\ -1 \\ 0 \end{bmatrix} e^{-b_1\sqrt{-p^2}d}.$$

In (25)–(28) the definitions

$$D_0 = |\mathbf{M}| = \frac{\Omega_1\Gamma_1\Gamma_2}{a_1 b_2}\left(1+\sqrt{\frac{\lambda_2}{\lambda_1}}\right)S_0 \tag{29a}$$

$$S_0 = Q^2+a_1 b_1 a_2 b_2+\Omega_1\Omega_2(a_1 b_2+a_2 b_1)-a_1 b_1 P_2^2-a_2 b_2 P_1^2 \tag{29b}$$

hold, where (29b) is the asymptotic thermoelastic Stoneley function for v, which has essentially the same form as its isothermal counterpart (Cagniard, 1962).

For $0 < y < d$ the asymptotic forms are

$$2\begin{bmatrix} u_{1x}^* \\ \dfrac{\sqrt{-p}}{\sqrt{p}}u_{1y}^* \\ \dfrac{1}{p}\theta_1^* \end{bmatrix} = \left(\frac{b_x}{\sqrt{-p^2}}\mathbf{M}_{1x}+\frac{b_y}{p}\mathbf{M}_{1y}^{(-)}+\frac{b_q}{\sqrt{-p^2}}\mathbf{M}_{1\theta}^{(-)}\right)\begin{bmatrix} e^{\sqrt{-\lambda_1}p(y-d)} \\ e^{a_1\sqrt{-p^2}(y-d)} \\ e^{b_1\sqrt{-p^2}(y-d)} \end{bmatrix}$$

$$+\left(\frac{b_x}{\sqrt{-p^2}}\mathbf{N}_{1x}+\frac{b_y}{p}\mathbf{N}_{1y}+\frac{b_q}{\sqrt{-p^2}}\mathbf{N}_{1\theta}\right)\begin{bmatrix} e^{-\sqrt{-\lambda_1 p}y} \\ e^{-a_1\sqrt{-p^2}y} \\ e^{-b_1\sqrt{-p^2}y} \end{bmatrix} \quad (30a)$$

$$\mathbf{M}_{1x}=\begin{bmatrix} 0 & -\dfrac{1}{a_1} & -b_1 \\ 0 & 1 & 1 \\ 0 & -\dfrac{\gamma_1}{a_1} & 0 \end{bmatrix}, \quad \mathbf{M}_{1y}^{(-)}=\begin{bmatrix} 0 & 1 & -1 \\ 0 & -a_1 & \dfrac{1}{b_1} \\ -\gamma_1 & \gamma_1 & 0 \end{bmatrix},$$

$$\Gamma_1\mathbf{M}_{1\theta}^{(-)}=\begin{bmatrix} 0 & \dfrac{1}{a_1} & 0 \\ 1 & -1 & 0 \\ -\Gamma_1\sqrt{\dfrac{p}{\lambda_1}} & 0 & 0 \end{bmatrix} \quad (30b)$$

$$\mathbf{N}_{1n}=\begin{bmatrix} 0 & -\dfrac{1}{\Gamma_1}B'_n & C'_n \\ 0 & -\dfrac{a_1}{\Gamma_1}B'_n & \dfrac{1}{b_1}C'_n \\ A'_n & -\dfrac{\gamma_1}{\Gamma_1}B'_n & 0 \end{bmatrix}, \quad \mathbf{N}_{1\theta}=\begin{bmatrix} 0 & \dfrac{1}{\Gamma_1}B'_b & C'_b \\ \dfrac{\sqrt{\lambda_1}}{\Gamma_1}A'_b & \dfrac{a_1}{\Gamma_1}B'_b & \dfrac{1}{b_1}C'_b \\ A'_b\sqrt{p} & 0 & 0 \end{bmatrix} \quad (30c,d)$$

where the subscript n denotes (x, y), the superscript $(-)$ on $(\mathbf{u}_1, \theta_1)$ is understood, and

$$\begin{bmatrix} A'_n \\ B'_n \\ C'_n \end{bmatrix}=\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M'_{22} & M'_{23} \\ 0 & M'_{32} & -M'_{33} \end{bmatrix}\begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix}, \quad \begin{bmatrix} A'_b \\ B'_b \\ C'_b \end{bmatrix}=\begin{bmatrix} M'_{11} & 0 & 0 \\ M_{21} & -M'_{22} & -M'_{23} \\ -M_{31} & M'_{32} & -M'_{33} \end{bmatrix}\begin{bmatrix} A_b \\ B_b \\ C_b \end{bmatrix}. \quad (31a,b)$$

Here the non-symmetric matrix $\mathbf{M}'$ is defined by

$$\begin{bmatrix} M'_{11} & M'_{12} & M'_{13} \\ M'_{21} & M'_{22} & M'_{23} \\ M'_{31} & M'_{32} & M'_{33} \end{bmatrix}=\begin{bmatrix} -\Gamma_2\Omega_1\left(1-\sqrt{\dfrac{\lambda_2}{\lambda_1}}\right) & M_{12} & M_{13} \\ M_{21} & \Gamma_1\left(P_2+\dfrac{a_2}{a_1}P_1\right) & -\Gamma_1\left(1+\dfrac{Q}{a_1 b_2}\right) \\ M_{31} & Q+a_2 b_1 & \dfrac{b_1}{b_2}P_2+P_1 \end{bmatrix}. \quad (32)$$

Finally, the asymptotic forms for $y > d$ are

$$2\begin{bmatrix} u_{1x}^* \\ \dfrac{\sqrt{-p}}{\sqrt{p}}u_{1y}^* \\ \dfrac{1}{p}\theta_1^* \end{bmatrix} = \left(\frac{b_x}{\sqrt{-p^2}}\mathbf{M}_{1x} + \frac{b_y}{p}\mathbf{M}_{1y}^{(+)} + \frac{b_q}{\sqrt{-p^2}}\mathbf{M}_{1\theta}^{(+)}\right)\begin{bmatrix} e^{\sqrt{-\lambda_1 p}(d-y)} \\ e^{a_1\sqrt{-p^2}(d-y)} \\ e^{b_1\sqrt{-p^2}(d-y)} \end{bmatrix}$$

$$+\left(\frac{b_x}{\sqrt{-p^2}}\mathbf{N}_{1x} + \frac{b_y}{p}\mathbf{N}_{1y} + \frac{b_q}{\sqrt{-p^2}}\mathbf{N}_{1\theta}\right)\begin{bmatrix} e^{-\sqrt{-\lambda_1 p}y} \\ e^{-a_1\sqrt{-p^2}y} \\ e^{-b_1\sqrt{-p^2}y} \end{bmatrix} \quad (33a)$$

$$\mathbf{M}_{1y}^{(+)} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -a_1 & \dfrac{1}{b_1} \\ \gamma_1 & -\gamma_1 & 0 \end{bmatrix}, \quad \Gamma_1\mathbf{M}_{1\theta}^{(+)} = \begin{bmatrix} 0 & \dfrac{1}{a_1} & 0 \\ -1 & 1 & 0 \\ -\Gamma_1\sqrt{\dfrac{p}{\lambda_1}} & 0 & 0 \end{bmatrix} \quad (33b)$$

where the superscript $(+)$ on $(\mathbf{u}_1, \theta_1)$ is understood.

5. Solution expressions by transform inversion

For a brief examination of solution behavior, attention is focused on solid 2 ($y < 0$): study of (25) in view of (28) indicates that $(\mathbf{u}_2, \theta_2)^*$ are linear combinations of generic functions of the transform variable p. These functions are

$$G_1^*(\alpha,\beta) = e^{\sqrt{-p^2}(\alpha y - \beta d)}, \quad G_2^*(\alpha,\beta) = \frac{\sqrt{p}}{\sqrt{-p}}G_1^*(\alpha,\beta) \quad (34a,b)$$

$$G_3^* = e^{\sqrt{-p}(\sqrt{\lambda_2}y - \sqrt{\lambda_1}d)}, \quad G_4^* = \frac{G_3^*}{\sqrt{-p}} \quad (34c,d)$$

$$G_5^*(\alpha) = e^{\sqrt{-p}(\alpha\sqrt{p}y - \sqrt{\lambda_1}d)}, \quad G_6^*(\alpha) = \frac{\sqrt{p}}{\sqrt{-p}}G_5^*(\alpha) \quad (34e,f)$$

$$G_7^*(\beta) = \frac{\sqrt{p}}{\sqrt{-p}}e^{\sqrt{-p}(\sqrt{\lambda_2}y - \beta\sqrt{p}d)} \quad (34g)$$

and

$$I_k^*(\alpha,\beta) = \frac{1}{p}G_k^*(\alpha,\beta)(k = 1,2), \quad I_3^* = \frac{1}{p}G_3^*, \quad I_k^*(\alpha) = \frac{1}{p}G_k^*(\alpha)(k = 5,6) \quad (35a\text{–}c)$$

where (α, β) are employed in this instance as positive real constants. In view of the branch cuts required for the asymptotic forms (24a), substitution of (35a) into the inversion operation (11b) allows a Bromwich contour that coincides with the entire $\mathrm{Im}(p)$-axis. Because $\alpha y - \beta d \leqslant 0$ for $y < 0$ while $\mathrm{Re}(\sqrt{-p^2}) \geqslant 0$ in the cut p-plane, exponential decay of the resulting integrand in (11b) is assured for $\mathrm{Re}(p) > 0$ when $x < 0$ and for $\mathrm{Re}(p) < 0$ for $x > 0$. Therefore, Cauchy theory can be used to change the integration path in (11b) to a contour that runs around the branch cut $\mathrm{Im}(p) = 0$, $\mathrm{Re}(p) > 0$ when $x < 0$, and around the branch cut $\mathrm{Im}(p) = 0$, $\mathrm{Re}(p) < 0$ when $x > 0$. The result is the real integral

$$G_1(\alpha, \beta) = -\frac{1}{\pi}\int_0^\infty e^{-p|x|} \sin(\alpha y - \beta d) p \, \mathrm{d}p \tag{36}$$

which, by use of a standard integral table (Gradshteyn and Ryzhik, 1965) can be evaluated as

$$G_1(\alpha, \beta) = -\frac{1}{\pi} \frac{\alpha y - \beta d}{x^2 + (\alpha y - \beta d)^2}. \tag{37}$$

By a similar procedure, the result

$$G_2(\alpha, \beta) = -\frac{1}{\pi} \frac{x}{x^2 + (\alpha y - \beta d)^2} \tag{38}$$

is obtained from (34b).

For (34c) the integrand branch cut is $\mathrm{Im}(p) = 0$, $\mathrm{Re}(p) > 0$, so that use of exponential decay and Cauchy theory give a zero result for $x > 0$ but the formula

$$G_3 = -\frac{1}{\pi}\int_0^\infty e^{px} \sin(\sqrt{\lambda_2} y - \sqrt{\lambda_1} d)\sqrt{p} \, \mathrm{d}p \tag{39}$$

for $x < 0$. The same integral table then gives

$$G_3 = -\frac{\sqrt{\lambda_2} y - \sqrt{\lambda_1} d}{\sqrt{\pi}(-x)^{3/2}} e^{\frac{1}{4x}(\sqrt{\lambda_2} y - \sqrt{\lambda_1} d)^2} \quad (x < 0) \tag{40}$$

and a similar procedure for (34d) gives

$$G_4 = \frac{1}{\sqrt{-\pi x}} e^{\frac{1}{4x}(\sqrt{\lambda_2} y - \sqrt{\lambda_1} d)^2} \quad (x < 0). \tag{41}$$

The form (34e) is in essence a product of forms similar to (34a,c). Therefore, the convolution theorem for bilateral Laplace transforms (van der Pol and Bremmer, 1950) can be used with (37) and (39) to produce the real integral

$$G_5(\alpha) = -\frac{\alpha\sqrt{\lambda_1} y d}{\pi^{3/2}} \int_0^\infty \frac{e^{-\frac{\lambda_1 d^2}{4t}}}{t^{3/2}} \frac{\mathrm{d}t}{(x+t)^2 + \alpha^2 y^2}. \tag{42}$$

Similarly, it can be shown that

$$G_6(\alpha) = -\frac{\sqrt{\lambda_1}d}{\pi^{3/2}}\int_0^\infty \frac{e^{-\frac{\lambda_1 d^2}{4t}}}{t^{3/2}}\frac{x+t}{(x+t)^2+\alpha^2 y^2}\,\mathrm{d}t \tag{43}$$

Inversion of (34g) follows by inspection from (43) as

$$G_7(\beta) = \frac{\sqrt{\lambda_2}y}{\pi^{3/2}}\int_0^\infty \frac{e^{-\frac{\lambda_2 y^2}{4t}}}{t^{3/2}}\frac{x+t}{(x+t)^2+\beta^2 d^2}\,\mathrm{d}t. \tag{44}$$

Finally, inversions of (35) follow essentially from indefinite integrations with respect to x of the functions (G_1, G_2, G_5, G_6):

$$I_1(\alpha,\beta) = -\frac{1}{\pi}\tan^{-1}\frac{x}{\alpha y-\beta d} \tag{45a}$$

$$I_2(\alpha,\beta) = -\frac{1}{\pi}\ln\sqrt{1+\frac{x^2}{(\alpha y-\beta d)^2}} \tag{45b}$$

$$I_3 = 2\,\mathrm{erf}\left(\frac{\sqrt{\lambda_1}d-\sqrt{\lambda_2}y}{2\sqrt{-x}}\right)\quad (x<0) \tag{45c}$$

$$I_5(\alpha) = -\frac{\sqrt{\lambda_1}d}{\pi^{3/2}}\int_0^\infty \frac{e^{-\frac{\lambda_1 d^2}{4t}}}{t^{3/2}}\tan^{-1}\frac{x+t}{\alpha y}\,\mathrm{d}t \tag{45d}$$

$$I_6(\alpha) = -\frac{\sqrt{\lambda_1}d}{\pi^{3/2}}\int_0^\infty \frac{e^{-\frac{\lambda_1 d^2}{4t}}}{t^{3/2}}\ln\sqrt{\frac{(x+y)^2+\alpha^2 y^2}{t^2+\alpha^2 y^2}}\,\mathrm{d}t. \tag{45e}$$

The results (45a–e) are, of course, valid to within an arbitrary additive constant. This result merely reflects the fact that the terms I_k appear in the displacement vector $\mathbf{u}_2$, which in the present steady-state analysis, can be determined only to within an arbitrary rigid-body motion.

Combining (36)–(45) with (25), (26) and (28) gives for $y<0$ the result

$$\begin{bmatrix} u_{2x} \\ u_{2y} \\ \theta_\theta \end{bmatrix} = \frac{1}{D_0}\begin{bmatrix} M_{11}U_{xx} & M_{11}U_{xy} & U_{xq} \\ M_{11}U_{yx} & M_{11}U_{yy} & U_{yq} \\ U_{qx} & U_{qy} & U_{qq} \end{bmatrix}\begin{bmatrix} b_x \\ b_y \\ b_q \end{bmatrix} \tag{46}$$

where additive arbitrary constants for (u_{2x}, u_{2y}) are understood, and the elements of matrix $\mathbf{U}$ are functions of (x, y) defined by

$$U_{xx} = -M_{33}\frac{\Gamma_1}{a_1}I_2(a_2,a_1) - M_{23}b_1 I_2(a_2,b_1) - M_{32}\frac{\Gamma_1}{a_1}I_2(b_2,a_1) - M_{22}b_1 I_2(b_2,b_1) \tag{47a}$$

$$U_{xy} = M_{33}\Gamma_1 I_1(a_2,a_1) - M_{23}I_1(a_2,b_1) + M_{32}\Gamma_1 I_1(b_2,a_1) - M_{22}I_1(b_2,b_1) \tag{47b}$$

$$U_{xq} = (M_{31}M_{23} - M_{21}M_{33})\frac{I_6(a_2)}{\sqrt{\lambda_1}} + (M_{31}M_{22} - M_{21}M_{32})\frac{I_6(b_2)}{\sqrt{\lambda_1}} + M_{11}M_{33}\frac{1}{a_1}I_2(a_2, a_1) - M_{11}M_{32}\frac{1}{a_1}I_2(b_2, a_1) \quad (47c)$$

$$U_{yx} = \frac{a_2}{a_1}M_{33}\Gamma_1 I_1(a_2, a_1) + a_2 b_1 M_{23} I_1(b_2, a_1) + \frac{\Gamma_1}{a_1 b_2}M_{32}\Gamma_1 I_1(b_2, a_1) + \frac{b_1}{b_2}M_{22}I_1(b_2, b_1) \quad (47d)$$

$$U_{yy} = -a_2 M_{33}\Gamma_1 I_2(a_2, a_1) + a_2 M_{23} I_2(a_2, b_1) - \frac{\Gamma_1}{b_2}M_{32}I_2(b_2, a_1) + \frac{1}{b_2}M_{22}I_2(b_2, b_1) \quad (47e)$$

$$U_{yq} = \frac{a_2}{\sqrt{\lambda_1}}(M_{21}M_{33} - M_{23}M_{31})I_5(a_2) + (M_{21}M_{32} - M_{31}M_{22})\frac{I_5(b_2)}{b_2\sqrt{\lambda_1}} - \frac{a_2}{a_1}M_{11}M_{33}I_1(a_2, a_1) + \frac{1}{a_1 b_2}M_{11}M_{32}I_1(b_2, a_1) + \sqrt{\frac{\lambda_2}{\lambda_1}}\frac{D_0}{M_{11}}I_3 \quad (47f)$$

$$U_{qx} = \frac{\Gamma_1\Gamma_2}{a_1}(M_{12}M_{33} - M_{13}M_{32})G_7(a_1) + \Gamma_2 b_1(M_{12}M_{23} - M_{13}M_{22})G_7(b_1) - \frac{\gamma_2\Gamma_1}{a_1}M_{11}M_{33}G_2(a_2, a_1) - \gamma_2 b_1 M_{11}M_{23}G_2(a_2, b_1) \quad (47g)$$

$$U_{qy} = \Gamma_2\Gamma_1(M_{11}M_{32} - M_{12}M_{33})G_5(a_1) + \Gamma_2(M_{12}M_{23} - M_{13}M_{22})G_5(b_1) + \gamma_2\Gamma_1 M_{11}M_{33}G_1(a_2, a_1) - \gamma_2 M_{11}M_{23}G_1(a_2, b_1) + \gamma_1\Gamma_2\frac{D_0}{M_{11}}G_3 \quad (47h)$$

$$U_{qq} = \frac{\Gamma_2}{\sqrt{\lambda_1}}\frac{D_0}{M_{11}}G_4. \quad (47i)$$

Examination of (46) in view of (47) and the formulas (38)–(45) reveals the effects of thermoelastic coupling: first of all, both the coefficients and the arguments of the functions (G_k, I_k) exhibit the thermoelastic parameters $(\varepsilon_1, \varepsilon_2, h_1, h_2)$ in the guise of the constants $(\Gamma_1, \Gamma_2, \lambda_1, \lambda_2, a_1, a_2, b_1, b_2)$. More importantly, the functions $(G_5, G_6, G_7, I_5, I_6)$ are in essence hybrids that combine behaviors normally associated with purely mechanical and thermal fields. That is, forms similar to (G_1, G_2, I_1, I_2) and (G_3, G_4, I_3) are archetype functions that appear in solutions to, respectively, the isothermal momentum balance equations and the thermal diffusion equation (Carslaw and Jaeger, 1959; Carrier and Pearson, 1988), while $(G_5, G_6, G_7, I_5, I_6)$ are convolutions that pair individuals from the different archetypes.

In regard to specific behavior, the functions (G_3, G_4, I_3) represent disturbances that occur only in the wake $(x < 0)$ of the moving sources. Thus, (46) shows that the (effective) thermal body source term (b_q) generates a temperature change θ_2 in its wake, while the mechanical source terms (b_x, b_y) create θ_2 everywhere.

As discussed in Boley and Weiner (1985), the effects of thermoelastic coupling may not be important when actual numerical calculations are performed. Examination of (40)–(44) and (45c–e) gives evidence of such a possibility here as well, because the very small [see (7)] thermoelastic characteristic lengths (h_1, h_2) appear in the argument denominator of decaying exponentials. However the magnitude of those arguments is also influenced by the distance of the observation point ($|y|$) and moving sources (d) from the welded interface.

Moreover, (24a,b) indicate that the aforementioned manifestations of thermoelastic parameters in certain constants occurs in such a way that the influence of the constants increases as the non-dimensionalized source speed (c_1, c_2) increases. Thus, the observations of Brock and Georgiadis (1997) that thermoelastic coupling becomes more important at higher source speed also holds for the present problem. It should be noted that analogous conclusions hold for dynamic quasi-brittle fracture (Brock, 1995, 1996) in terms of crack propagation speed. Therefore, such coupling may be more than just formally important.

6. Identification of the critical source speed

Despite the construction of solutions, the source speed v is so far restricted by the inequality (8). However, the presence of the theremoelastic Stoneley function (22a) and its asymptotic counterpart (29b) in the denominator of, respectively, the transform and asymptotic solutions suggest that a stricter definition of critical speed is needed. Brock (1997) has studied the asymptotic form S_0 given in (29b) and found that it behaves much like its isothermal counterpart (Cagniard, 1962). In particular, S_0 will exhibit the real zeroes $v = \pm v_S$ if the thermoelastic properties of the welded materials are such that

$$s_{r1}^2(\kappa_1+\kappa_2-s_{r1}^2)^2+[\kappa_1\kappa_2\sqrt{s_{r1}^2-s_{\varepsilon 1}^2}-(\kappa_1-s_{r1}^2)^2\sqrt{s_{r1}^2-s_{\varepsilon 2}^2}]\sqrt{s_{r1}^2-s_{r2}^2} > 0 \tag{48a}$$

when $s_{r1} > s_{r2}$ or

$$s_{r2}^2(\kappa_1+\kappa_2-s_{r2}^2)^2+[\kappa_1\kappa_2\sqrt{s_{r2}^2-s_{\varepsilon 2}^2}-(\kappa_2-s_{r2}^2)^2\sqrt{s_{r2}^2-s_{\varepsilon 1}^2}]\sqrt{s_{r2}^2-s_{r1}^2} > 0 \tag{48b}$$

when $s_{r2} > s_{r1}$, where cf (23)

$$\kappa_1 = \frac{\mu_1 s_{r1}^2}{2(\mu_1-\mu_2)}, \quad \kappa_2 = \frac{\mu_2 s_{r2}^2}{2(\mu_2-\mu_1)}, \quad s_{\varepsilon 1} = \frac{s_{d1}}{\sqrt{1+\varepsilon_1}}, \quad s_{\varepsilon 2} = \frac{s_{d2}}{\sqrt{1+\varepsilon_2}}. \tag{49}$$

The terms ($s_{\varepsilon 1}, s_{\varepsilon 2}$) are the asymptotic thermoelastic values of the dilatational slownesses. Moreover, v_S, which corresponds to an asymptotic thermoelastic Stoneley interface wave speed, lies in the range $0 < v_S < \min(v_{r1}, v_{r2})$. By means of product-splitting techniques (Noble, 1958), Brock (1997) has obtained the result

$$v_S = G_0\sqrt{\frac{(4\kappa_1-s_{\varepsilon 1}^2-s_{r1}^2)(s_{\varepsilon 2}^2+s_{r2}^2-4\kappa_2)}{2|\kappa_2 s_{\varepsilon 1}-\kappa_1 s_{\varepsilon 2}|\,|\kappa_2 s_{r1}-\kappa_1 s_{r2}|}}. \tag{50}$$

Table 1

	v_{R1} (m/s)	v_{R2} (m/s)	v_S (m/s)
Aluminum(1) Steel(2)	2857	2842	3026
Aluminum(1) Titanium(2)	2857	2826	3027

For the slowness combinations $s_{r2} > s_{\varepsilon 2} > s_{r1} > s_{\varepsilon 1}$, the dimensionless constant G_0 has the form

$$\ln G_0 = -\frac{1}{\pi}\left(\int_{s_{\varepsilon 1}}^{s_{r1}} \phi_1 \frac{\mathrm{d}t}{t} + \int_{s_{r1}}^{s_{\varepsilon 2}} \phi_{12} \frac{\mathrm{d}t}{t} + \int_{s_{\varepsilon 2}}^{s_{r2}} \phi_2 \frac{\mathrm{d}t}{t}\right). \tag{51}$$

Here the ϕ-functions lie in the range $(0, \pi/2)$ and are defined by

$$\phi_1 = \tan^{-1} \xi_1 \frac{-t^2\eta_1\xi_2\eta_2 + \kappa_1\kappa_2\eta_2 - \eta_1 T_2^2}{-\kappa_1\kappa_2\xi_2\eta_1 + \xi_2\eta_2 T_1^2 + t^2 T_{12}^2} \tag{52a}$$

$$\phi_{12} = \tan^{-1} \frac{\kappa_1\kappa_2(\xi_1\eta_2 + \xi_2\eta_1)}{-t^2\xi_1\eta_1\xi_2\eta_2 + \xi_2\eta_2 T_1^2 - \xi_1\eta_1 T_2^2 + t^2 T_{12}^2} \tag{52b}$$

$$\phi_2 = \tan^{-1} \eta_2 \frac{t^2\xi_1\eta_1\xi_2 + \kappa_1\kappa_2\xi_1 - \xi_2 T_1^2}{\kappa_1\kappa_2\xi_2\eta_1 - \xi_1\eta_1 T_2^2 + t^2 T_{12}^2} \tag{52c}$$

where cf (13a), (23), (24b),

$$\xi = \sqrt{|t^2 - s_\varepsilon^2|}, \quad \eta = \sqrt{|t^2 - s_r^2|}, \quad T_1 = \kappa_1 - t^2, \quad T_2 = \kappa_2 - t^2, \quad T_{12} = \kappa_1 + \kappa_2 - t^2 \tag{53}$$

and the subscripts $(1, 2)$ are understood in the equations for (ξ, η). The other five (excluding equalities as special cases) possible combinations of material slownesses give similar forms.

Asymptotic thermoelastic Rayleigh speeds (v_{R1}, v_{R2}) exist for solids, and so we also consider them to be candidates for critical speed. By the same process used to obtain (50), it can be shown that such speeds always exist, and are given by the general formula

$$v_R = v_r G_0 \sqrt{2\left(1 - \frac{1}{m^2(1+\varepsilon)}\right)} \tag{54}$$

where $0 < v_R < v_r$ and, in this case, the dimensionless constant G_0 is given by

$$\ln G_0 = -\frac{1}{\pi}\int_{s_\varepsilon}^{s_r} \tan^{-1} \frac{4t^2\sqrt{t^2 - s_\varepsilon^2}\sqrt{s_r^2 - t^2}}{(2t^2 - s_r^2)^2} \frac{\mathrm{d}t}{t}. \tag{55}$$

In (54) and (55), the subscripts $(1, 2)$ are understood. In view of (7), it is clear from (51) and (54) that thermoelastic coupling has only a minor effect on the value of the asymptotic Stoneley interface and Rayleigh speeds. In Table 1 values of v_S are given for two combinations of thermoelastic materials for which it exists, along with the corresponding values of (v_{R1}, v_{R2}). It is seen that v_S for these cases in fact exceeds the Rayleigh speeds. However, to ensure non-critical behavior due to source speed, we replace the restriction (8) with the relation

$$0 < v < \min(v_S, v_{R1}, v_{R2}) \tag{56}$$

with the understanding that v_S is dropped when neither (48a) nor (48b) is satisfied. It should be noted that (56) does not preclude the possibility of very rapid source motion.

7. Some observations

This article extended the study of thermomechanical source motion over the surfaces of thermoelastic half-spaces to the problem of buried thermomechanical sources moving parallel to the interface of two welded dissimilar thermoelastic solids. The momentum balance and thermal diffusion equations were treated in their coupled form, and the sources were line loads that appeared as body force terms in these equations. These loads moved at a constant subcritical speed so that the problem analysis was 2-D and steady-state.

Exact bilateral transform solutions were derived, and then asymptotic forms of these solutions were inverted analytically to give expressions for the displacements and temperature change in one of the solids. These expressions were in principle valid for large distances from the moving sources. However, the scaling distances were the thermoelastic characteristic lengths of the solids which, being of $O(10^{-4})$ μm, guaranteed that the expressions were actually robust.

These expressions showed the effects of thermoelastic coupling. In particular, the thermal body force term was seen to manifest itself in the solution in an effective thermal source term that also exhibited one component of the mechanical body force. Then, thermoelastic constants were seen to modify both the coefficients and the arguments of the mathematical functions that comprised the solution expressions. Moreover, while some of the functions themselves were archetypes that arise either in isothermal mechanical problems or purely thermal diffusion problems, other functions were in essence hybrids of the two archetypes. Moreover, the thermoelastic parameters of the two solids occurred in combinations with the source speed in such a manner that their influence increased with the source speed, thereby suggesting that thermoelastic coupling is important at high subcritical source speeds.

The critical speed was taken to be the minimum of the two asymptotic thermoelastic Rayleigh speeds in the two solids and—when it exists—the asymptotic thermoelastic Stoneley interface wave speed. This restriction avoids the possibility of unbounded behavior at the latter speed. The conditions for which the Stoneley speed exists were given, as well as an exact expression for it and the Rayleigh values. Numerical calculations with these formulas for specific cases suggested that the Stoneley value exceeds the Rayleigh values.

In summary, then, the response of solids formed from welded dissimilar thermoelastic materials to moving thermomechanical body forces is influenced at higher source speeds by thermoelastic coupling.

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